Winter semester 2017/18
Prof. Dr. S. Held
Dr. U. Brenner

Research Institute for Discrete Mathematics
University of Bonn

Linear and Integer Optimization

Exercise Sheet 3

Exercise 3.1:

(a) Prove the generalized Farkas lemma (Lemma 3.3): Let $A \in \mathbb{K}^{m_1 \times n_1}, B \in \mathbb{K}^{m_1 \times n_2}$, $C \in \mathbb{K}^{m_2 \times n_1}, D \in \mathbb{K}^{m_2 \times n_2}, a \in \mathbb{K}^{m_1}$ and $b \in \mathbb{K}^{m_2}$. Then exactly one of the following systems has a solution:

$$\left\{\begin{array}{rcl} Ax + By & \leq & a \\ Cx + Dy & = & b \\ x & \geq & 0 \end{array}\right\} \quad \dot{\bigvee} \quad \left\{\begin{array}{rcl} u^\intercal A + v^\intercal C & \geq & 0^\intercal \\ u^\intercal B + v^\intercal D & = & 0^\intercal \\ u & \geq & 0 \\ u^\intercal a + v^\intercal b & < & 0. \end{array}\right\}.$$

(b) Let (P) be a linear program of the form $\min\{c^\intercal x : Ax \leq b\}$. Prove that the dual of the dual is equivalent to (P) (Lemma 3.11).

(2+2 points)

Exercise 3.2: Prove the following theorems of alternatives:

(a) $(\exists x : Ax \leq c,\ Ax \neq c)$

$$\dot{\vee}\ \Big(\exists y : (A^T y = 0,\ c^T y = -1,\ y \geq 0) \vee (A^T y = 0,\ c^T y \leq 0,\ y > 0)\Big)$$

(3 Points)

(b) $(\exists x : Ax > 0,\ Cx \geq 0,\ Dx = 0)$

$$\dot{\vee}\ (\exists u, v, w : u, v \geq 0,\ u \neq 0,\ A^T u + C^T v + D^T w = 0)$$ (3 Points)

Exercise 3.3: Consider the following linear program $\min\{c^\intercal x : Ax = b\}$. Show that it either does not have a solution, is unbounded, or that all feasible solutions are optimal. Does this statement hold if we additionally require $x \geq 0$? (3 Points)

Exercise 3.4: Let P be a polyhedron. Show that the problem of finding a largest ball that is contained in P can be written as a linear program.

(3 Points)

Submission deadline: Thursday, November 2, 2017, before the lecture (in groups of 2 students).