Summer term 2026
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Linear and Integer Optimization

Exercise Sheet 12

Exercise 12.1: Consider the following problem: We are given a directed graph G and nodes $s, t \in V(G)$ with $s \neq t$. Moreover, we are given integral mappings $l, u : E(G) \to \mathbb{Z}$ such that $l(e) \leq u(e)$ for all $e \in E(G)$. The task is to find a mapping $f : E(G) \to \mathbb{R}$ with $l(e) \leq f(e) \leq u(e)$ for all edges $e \in E(G)$ and $\sum_{e \in \delta^-_G(v)} f(e) = \sum_{e \in \delta^+_G(v)} f(e)$ for all $v \in V(G) \setminus \{s, t\}$ such that $\sum_{e \in \delta^+_G(s)} f(e) - \sum_{e \in \delta^-_G(s)} f(e)$ is maximized. This problem generalizes the max-flow problem. Show that there is always an integral optimum solution and show that the value of a maximum solution equals

$$\min \left\{ \sum_{e \in \delta^+_G(X)} u(e) - \sum_{e \in \delta^-_G(X)} l(e) \mid X \subseteq V(G) \setminus \{t\}, s \in X \right\}.$$

(4 points)

Exercise 12.2: Let A and B be integral matrices, a and d integral column vectors, and b and c integral row vectors of appropriate sizes. Show that the following compositions $\oplus_1$, $\oplus_2$, and $\oplus_3$ preserve total unimodularity, i.e. assume that the two matrices on the left-hand side are totally unimodular and show the same for the matrix on the right-hand side:

(a) $A \oplus_1 B := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

(b) $\begin{bmatrix} A & a \end{bmatrix} \oplus_2 \begin{bmatrix} b \\ B \end{bmatrix} := \begin{bmatrix} A & ab \\ 0 & B \end{bmatrix}$

(c) $\begin{bmatrix} A & a & a \\ c & 0 & 1 \end{bmatrix} \oplus_3 \begin{bmatrix} 1 & 0 & b \\ d & d & B \end{bmatrix} := \begin{bmatrix} A & ab \\ dc & B \end{bmatrix}$ (1+3+4 points)

Exercise 12.3: Let $S := \{x \in \mathbb{Z}^2_+ \ : \ 4x_1 + x_2 \leq 28, x_1 + 4x_2 \leq 27, x_1 - x_2 \leq 1\}$. Describe the facets of $\mathrm{conv}(S)$ by (iteratively) applying Gomory-Chvátal cuts. (A few cuts are sufficient, you may first find the facets geometrically) (4 points)

Exercise 12.4:

(a) Give an example of a polyhedron P with $P_I \neq P^{(i)}$ for all $i \in \mathbb{N}$.

(b) Let P the convex hull of the three points $(0, 0)$, $(1, 0)$, and $(\frac{1}{2}, k)$ in $\mathbb{R}^2$, where $k \in \mathbb{N}$. Prove that $P^{(2k-1)} \neq P_I$, but $P^{(2k)} = P_I$. (2+2 Points)

Submission deadline: Tuesday, July 14, 2026, 16:00, via eCampus (in groups of at most 3 students).