Summer term 2026
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Linear and Integer Optimization

Exercise Sheet 4

Exercise 4.1: Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron. Moreover, let $P^* := \{y \in \mathbb{R}^n \mid y^t x \leq 1 \text{ for all } x \in P\}$ and $P^0 := \{y \in \mathbb{R}^n \mid y^t x \leq 0 \text{ for all } x \in P\}$.

(a) Show that P^* is a polyhedron.

(b) Prove that $(P^*)^* = P$ if and only if $b \geq 0$.

(c) In addition, assume $b = 0$. Show that $P^* = P^0$ and prove that P^0 is the convex cone generated by the rows of A. (2+3+2 points)

Exercise 4.2: Let $H = (V, E)$ be a hypergraph, so V is a finite set of nodes and $E \subseteq 2^V$. Assume that you are given $F \subseteq V$ and $x, y : F \to \mathbb{R}$.

(a) Describe the following problem as a linear program. We ask for an extension $x, y : V \setminus F \to \mathbb{R}$ such that

$$\sum_{e \in E} \left(\max_{v \in e} x(v) - \min_{v \in e} x(v) + \max_{v \in e} y(v) - \min_{v \in e} y(v) \right)$$

is minimized.

(b) Dualize the LP from (a) and show that the dual LP is equivalent to a MIN-COST FLOW PROBLEM (see below). (3+3 points)

Hint: In the MIN-COST FLOW PROBLEM you are given a directed graph G, edge capacities $u : E(G) \to \mathbb{R}_{>0}$, edge costs $c : E(G) \to \mathbb{R}$ and values $b : V(G) \to \mathbb{R}$ with $\sum_{v \in V(G)} b(v) = 0$. We ask for a mapping $f : E(G) \to \mathbb{R}_{\geq 0}$ with $\sum_{e \in \delta_G^+(v)} f(e) - \sum_{e \in \delta_G^-(v)} f(e) = b(v)$, such that $\sum_{e \in E} f(e) \cdot c(e)$ is minimized.
Remark: This is a relaxation of the placement problem in chip design. The vertices correspond to connected modules that must be placed minimizing the length of all interconnects (hyperedges). Vertices in F are preplaced. The problem becomes much harder when requiring disjointness of the modules.

Exercise 4.3: Consider the following linear system of inequalities:

$$\begin{aligned} x_1 + x_2 &\leq 6 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Sketch the solution space, transform the system by adding slack variables x_3, x_4, x_5 to the form $Ax = b, x \geq 0$, and determine all bases and the corresponding basic solutions. Are there any degenerate or infeasible basic solutions? If so, which ones? (3 points)

p.t.o.

Exercise 4.4: Consider a linear program in appropriate form for the SIMPLEX ALGORITHM, i.e. $\max\{c^t x \mid Ax = b, x \geq 0\}$ such that $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$ and $Ax = b$ is feasible. Prove or disprove the following statements:

(a) A variable that has just entered the basis can leave the basis in the next iteration.

(b) A variable that has just left the basis can enter the basis in the next iteration.

(c) If x is unique optimum basic solution and $\tilde{x}$ a second best basic solution with strictly smaller solution value then x can be computed from $\tilde{x}$ by exchanging one basic variable.

(d) If no basic solution is degenerated and the LP is feasible and bounded then there is a unique optimum solution. (1+1+1+1 points)

Submission deadline: Tuesday, May 12, 2026, 16:00, via eCampus (in groups of at most 3 students).