

## Linear and Integer Optimization

### Exercise Sheet 2

**Exercise 2.1:** Two sets  $X, Y \subseteq \mathbb{R}^n$  are called *strictly separable*, if there is a vector  $a \in \mathbb{R}^n \setminus \{0\}$  such that  $a^t x < b$  for all  $x \in X$  and  $a^t y > b$  for all  $y \in Y$ . Prove or disprove the following statement: Any two disjoint closed convex sets are strictly separable. (3 points)

**Exercise 2.2:** The convex hull of a set  $Y \subseteq \mathbb{K}^n$  is defined as  $\text{conv}(Y) := \left\{ \sum_{i=1}^k \lambda_i x_i : k \in \mathbb{N}, 0 \leq \lambda_i \leq 1, x_i \in Y \text{ for } 1 \leq i \leq k; \sum_{i=1}^k \lambda_i = 1 \right\}$ .

- (a) Prove that for each set  $X \subseteq \mathbb{R}^n$  the set  $\text{conv}(X)$  is the smallest convex set containing  $X$ .
- (b) Prove that any set  $X \subseteq \mathbb{R}^n$  with  $|X| > n + 1$  can be decomposed into subsets  $X_1$  and  $X_2$  such that  $\text{conv}(X_1) \cap \text{conv}(X_2) \neq \emptyset$ . (2+5 points)

**Exercise 2.3:** Consider the following problem: Let  $G$  be a directed graph with edge weights  $w : E(G) \rightarrow \mathbb{R}$ . We are looking for a directed cycle  $C$  in  $G$  with  $|V(G)|$  edges such that  $\sum_{e \in E(C)} w(e)$  is minimized. Consider the following mixed-integer linear programming problem:

$$\begin{aligned} \min \quad & \sum_{e \in E} x_e w(e) \\ \text{s.t.} \quad & \sum_{e \in \delta_G^+(v)} x_e = 1 \quad \text{for all } v \in V(G) \\ & \sum_{e \in \delta_G^-(v)} x_e = 1 \quad \text{for all } v \in V(G) \\ & u_v \geq 1 \quad \text{for all } v \in V(G) \\ & u_v \leq |V| \quad \text{for all } v \in V(G) \\ & x_e \in \{0, 1\} \quad \text{for all } e \in E(G) \end{aligned}$$

Add  $O(|V(G)|^2)$  constraints to this formulation (using the currently superfluous variables  $u_v$ ), so that a mixed-integer LP is obtained whose solution directly yields an optimal cycle, if one exists. (5 points)

**Exercise 2.4:** An *interior point* of a polyhedron  $P \subseteq \mathbb{R}^n$  is a vector  $x \in P$  that does not lie in any face of  $P$  that is distinct from  $P$  itself (note that this definition differs from the concept of interior points in the topological sense). Prove or disprove: Every non-empty polyhedron contains an interior point. (5 points)

**Submission deadline:** Tuesday, April 28, 2026, 16:00, via eCampus (in groups of at most 3 students).